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ABSTRACT

A method of calculating the constants of a Maxwell model from the constants of the corresponding Voigt model is given. The derivation is based on residue theory and the partial fraction expansion of the transfer functions. As an example, constants for a 10-element Maxwell model are derived from the constants of a 10-element Voigt model used to approximate the mechanical properties of polyisobutylene at 25°C. Formulas for the creep and relaxation functions are also given.

INTRODUCTION

Voigt and Maxwell models are frequently used in analyzing data from mechanical tests of viscoelastic materials. A generalized Voigt model is generally more convenient for representing creep and complex compliance data, whereas the generalized Maxwell model is usually used for relaxation and complex modulus data. To compare the results of different types of tests, a method is required for expressing the constants of a given type of model in terms of constants of the other type. Alfrey[1] has shown that relations between models consisting of three or four elements may be derived by elementary algebraic methods. When the models consist of a larger number of elements, the relations among the constants become very complicated, and a method based on residue theory is more effective.

On several occasions, the author has been asked to discuss this problem and outline a method of calculation that would be suitable for machine calculation. For this reason, a detailed analysis of the problem is given together with a numerical example. Gross has outlined a method based directly on the theory of limits.* However, his argument is somewhat difficult to follow. It is hoped that this report will clarify the analytical relations between Voigt and Maxwell models.

ANALYSIS

Springs and dashpots are the fundamental elements of a viscoelastic model. In the linear theory of viscoelasticity, it is assumed that the stress in the springs is proportional to the strain, while the stress, σ in the dashpots is proportional to the rate of strain, $\frac{d\epsilon}{dt}$. The stress-strain law for the entire model may then be expressed as a differential equation with constant coefficients:

$$p_0\sigma + p_1 \frac{d\sigma}{dt} + \dots + p_n \frac{d^n\sigma}{dt^n} = q_0\epsilon + q_1 \frac{d\epsilon}{dt} + \dots + q_m \frac{d^m\epsilon}{dt^m} \tag{1}$$

Viscoelastic models differing in configuration are considered equivalent if they are governed by the same stress-strain law.

* Pages 64-65, Reference 3

In Eq. (1), the low order terms govern the behavior at long times, while the high order terms govern the behavior at short times. If there is long-term viscous flow, $q_0 = 0$; otherwise $q_0 > 0$. If the material exhibits instantaneous elastic response, $m = n$; if the elastic response is retarded, $m = n + 1$. These considerations lead to the four classes of models illustrated in Fig. 1. Voigt and Maxwell representations are shown for each class. These models adhere to the topological relations given by Alfrey* and are numbered according to Table IV of Gross's monograph.** In this figure, the E_j represent the spring constants and the η_j represent constants for the dashpots.

The limiting properties of each class and the appropriate stress-strain law are shown in Table I. It is clear that constants for a given model can be derived from constants of another model only if both models belong to the same class and have the same number of elements.

Table I. Properties of Viscoelastic Models According to Class

Class	Elastic Response	Long Term Viscous Flow	Form of Eq. (1)
I	Instantaneous	Absent	$m = n;\ q_0 > 0$
II	Instantaneous	Present	$m = n;\ q_0 = 0$
III	Retarded	Absent	$m = n + 1;\ q_0 > 0$
IV	Retarded	Present	$m = n + 1;\ q_0 = 0$

* Page 545, Reference 1

** Page 62, Reference 3

FIG. 1. EQUIVALENT VISCOELASTIC MODELS CLASSIFIED ACCORDING TO RESPONSE

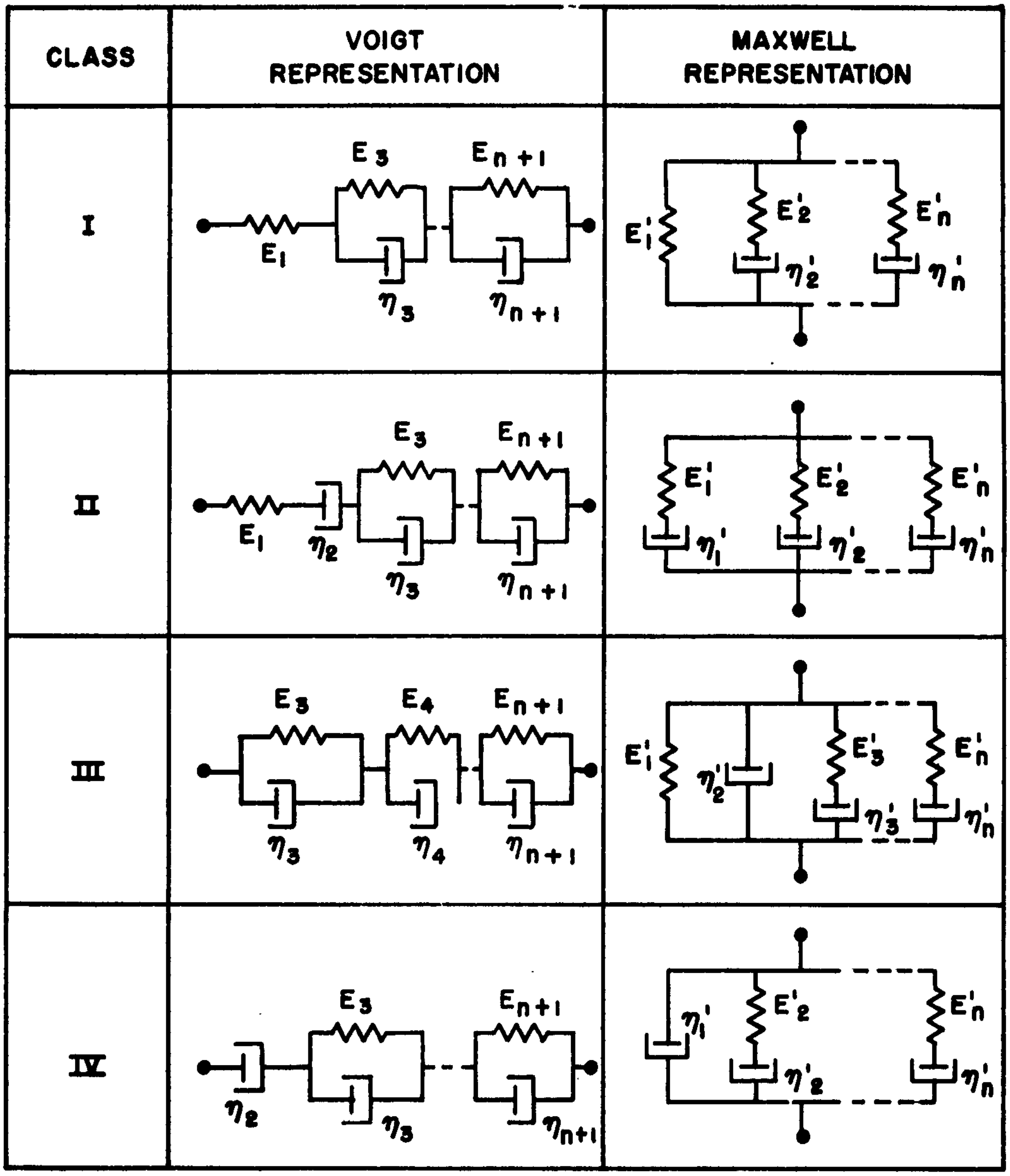

Table II. General Form of Viscoelastic Functions**

Viscoelastic Function	General Form
Complex Compliance	$J^*(i\omega) = \sum_{j=3}^{n+1} \frac{1}{E_j + i\omega\eta_j} + J_1^*(i\omega)$
Transfer Function	$J(s) = \sum_{j=3}^{n+1} \frac{C_j}{s - \lambda_j} + J_1(s)$
Creep Function, $t > 0$	$J(t) = \sum_{j=3}^{n+1} (1 - e^{\lambda_j t}) + \epsilon_1(t)$
Complex Modulus	$G^*(i\omega) = \sum_{j=K}^{n} \frac{i\omega\eta'_j E'_j}{E'_j + i\omega\eta'_j} + G^*_1(i\omega)$
Transfer Function	$G(s) = \sum_{j=K}^{n} \frac{C'_j s}{s - \mu_j} + G_1(s)$
Relaxation Function, $t > 0$	$G(t) = \sum_{j=K}^{n} C'_j e^{\mu_j t} + \sigma_1(t)$

Table III. Remainder Term for Each Class of Model**

Remainder Terms, Viscoelastic Functions	Class of Model I	II	III	IV
$J^*_1(i\omega)$	$\frac{1}{E_1}$	$\frac{1}{E_1} + \frac{1}{i\eta_2\omega}$	0	$\frac{1}{i\eta_2\omega}$
$J_1(s)$	A	$A + \frac{B}{s}$	0	$\frac{B}{s}$
$\epsilon_1(t)$	A	A + Bt	0	Bt
$G^*_1(i\omega)$	E'_1	0	$E'_1 + iw\eta'_2$	$i\omega\eta'_2$
$G_1(s)$	A'	0	A' + B's	B's
$\sigma_1(t)$	A'	0	$A' + B'\delta(t)$	$B'\delta(t)$
K	2	1	3	2

** In Tables II and III, ω is the circular frequency, s the transform variable in the Laplace transform, and t is real time.

The Voigt and Maxwell representations lead to expansions in terms of partial fractions. Expressions for the creep and relaxation functions are written in terms of the constants occurring in these expansions. The general forms of the various viscoelastic functions are shown in Table II. A remainder term depending on the class of model must be added to the general form in order to obtain the complete expression; these remainder terms are given in Table III.

In these tables, the λ_j are roots of the polynomial equation

$$q_0 + q_1 s + \dots + q_m s^m = 0 \tag{2}$$

while the μ_j are the roots of the equation

$$p_0 + p_1 s + \dots + p_n s^n = 0 \tag{3}$$

The delta function $\delta(t)$ is required in the expression for the relaxation function for Classes III and IV,* since the models in these classes cannot yield instantaneously to a load of finite magnitude.

Transformation of models can be carried out only for models belonging to the same class.** Consider for instance, the generalized Voigt model and the corresponding Maxwell model of Class II, and suppose the constants for the Voigt model are known. The roots μ_1, μ_2 ... of the equation

$$A + \frac{B}{s} + \sum \frac{C_j}{s - \lambda_j} = 0 \tag{4}$$

are calculated by some method of successive approximation. These roots are real and negative, and are interlaced with the roots λ_1, λ_2 ... .

The constants C'_j are obtained by a limiting process.

$$J(s) = A + \frac{B}{s} + \sum \frac{C_j}{s - \lambda_j}$$

$$G(s) = \sum \frac{C'_j}{s - \mu_j}$$

* Page 38, Reference 2

** Pages 61 and 62, Reference 3

so that

$$C'_j = \lim_{s \to \mu_j} (s - \mu_j) \frac{G(s)}{s}$$

But $$G(s) = \frac{1}{J(s)}$$

$$(s - \mu_j)\, G(s) = \frac{s - \mu_j}{J(s)}$$

$$= \frac{s - \mu_j}{s(A + \frac{B}{s} + \sum \frac{C_j}{s - \lambda_j})}$$

and

$$C'_j = \lim_{s \to \mu_j} \left\{ \frac{s - \mu_j}{s(A + \frac{B}{s} + \sum \frac{C_j}{s - \lambda_j})} \right\}$$

The indeterminate form on the right is evaluated by L'Hospital's rule.* The final result is

$$C'_j = \frac{1}{A + \sum_{k=1} \frac{(-C_k \lambda_k)}{(\mu_j - \lambda_k)^2}} \tag{5}$$

To go from the Maxwell model to the Voigt model, the roots λ_1, λ_2 ... of the equation

$$\sum \frac{C'_j}{s - \mu_j} = 0 \tag{6}$$

are calculated in the manner indicated above. The coefficients C_j are calculated from the formula

$$C_j = \frac{1}{\sum_{k=1} \frac{-\mu_k C'_k}{(\lambda_j - \mu_k)^2}} \quad . \tag{7}$$

* This rule is given in standard texts on the calculus. See for instance, Page 346, Reference 4.

The constants A and B are derived from the relation

$$G(s) \cdot J(s) = 1$$

$$\left\{\sum \frac{C'_j s}{s - \mu_j}\right\}\left\{A + \frac{B}{s} + \sum \frac{C_j}{s - \lambda_j}\right\} = 1$$

or

$$\left\{\sum \frac{C'_j}{s - \mu_j}\right\}\left\{As + B + \sum \frac{sC_j}{s - \lambda_j}\right\} = 1 \quad .$$

Let $s \to 0$; then

$$B = \frac{1}{\sum \frac{C'_j}{- \mu_j}} \quad . \tag{8}$$

The value of A is found by allowing s to become infinite.

$$A = \frac{1}{\sum C'_j} \quad . \tag{9}$$

Formulas for the other three classes of models may be obtained in the same manner.

AN EXAMPLE

Consider a generalized Voigt model used to approximate the dynamic data for polyisobutylene at 25°C. The fitting constants were calculated by a semi-graphical method described in Reference 5. The calculated values of the complex compliance generally agree with the measured values within the limits of experimental error over the entire frequency range.

COMPLEX COMPLIANCE OF POLYISOBUTYLENE

$25^{\circ}C$

Frequency cps	Measured Values (Ref 6) $J'(\omega)$ cm^2/dynes	Measured Values (Ref 6) $J''(\omega)$ cm^2/dynes	Calculated Values (Ref 5) $J'(\omega)$ cm^2/dynes	Calculated Values (Ref 5) $J''(\omega)$ cm^2/dynes
30	204	102	203	100
40	184	93.6	186	100
45	182	107	178	100
60	156	98.0	159	99.6
80	137	97.8	138	97.7
100	119	95.2	122	94.5
140	98.4	88.1	99.3	86.7
200	79.0	77.9	79.3	76.0
280	65.3	65.0	64.4	66.2
400	51.2	56.4	51.4	56.9
600	38.5	46.9	38.9	47.3
800	32.1	41.0	31.7	40.8
1000	27.1	36.4	27.1	36.3
1400	20.7	30.1	21.3	30.3
2000	15.7	26.3	16.2	25.0
2800	12.1	20.5	12.1	20.5
4000	8.94	16.6	8.64	16.1

The calculated values tabulated above are based on the following model constants.

MODEL CONSTANTS: 10-ELEMENT VOIGT MODEL

j	A (cm^2/dyne) x 10^{-9}	B (cm^2/dyne sec) x 10^{-9}	C_j (cm^2/dyne sec) x 10^{-9}	λ_j rad/sec
	3.365	4.756		
1			13110	-165
2			71380	-610
3			122600	-3000
4			264500	-15000

The values of the μ_j for the corresponding Maxwell model were found by Newton's method. The fitting constants C'_j were then obtained from Equation 5. The results are given below.

MODEL CONSTANTS: 10-ELEMENT MAXWELL MODEL

j	C'_j (dynes/cm^2) x 10^9	μ_j rad/sec
1	0.003498	-17.52
2	0.0009847	-222.5
3	0.004164	-1429
4	0.01061	-7611
5	0.2780	-151400

The roots λ_j and μ_j are interlaced along the negative real axis in the manner specified by the general theory; moreover the constants A, B, C_j, and C'_j are all positive. The values of the individual springs and dashpots derived from these constants will be positive, as required by the physical theory.

The relaxation function is expressed in terms of the constants A, B, C_j, and λ_j for the generalized Voigt model. The creep function, on the other hand, is expressed in terms of the constants C'_j and μ_j of the generalized Maxwell model. The formulas given in this note afford a simple method of calculating the mechanical response of linear viscoelastic materials under a variety of loading conditions after a model corresponding to a particular type of loading has been established.

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REFERENCES

1. Alfrey, T., Jr. Mechanical Behavior of High Polymers New York, Interscience Publishers, 1948.

2. Bland, D. R. The Theory of Linear Viscoelasticity New York, Pergamon Press, 1960.

3. Gross, B. Mathematical Structure of the Theories of Viscoelasticity Paris, Herman and Cie, 1953.

4. Smail, Lloyd L. Calculus New York, Appleton-Century-Crofts, Inc., 1949.

5. Goldberg, W. and Dean, N. W. Determination of Viscoelastic Model Constants from Dynamic Mechanical Properties of Linear Viscoelastic Materials, Aberdeen Proving Ground, BRL Report No. 1180, November 1962.

6. Fitzgerald, E. R.; Grandine, L. D.; and Ferry, J. D. Dynamic Mechanical Properties of Polyisobutylene, Journal of Applied Physics, 24: 650-655, 1953.

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